A Gentle Introduction to Category Theory

Alistair Wallis

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Introduction
- Introduction

Classical Category Theory
- Definitions
- Examples
- Types of arrows
- Functors
- Natural transformations

Comments
- Advantages and Disadvantages

Categories as monoids
- Motivation
- Basics

Introduction

- In 1940's and 50's mathematicians were very interested in generalising ideas further than before
- Examples: abstract differential manifolds and abstract varieties
- Universal algebra
- In this backdrop, category theory invented / described

Definition of a category

Definition

A *category* C consists of a collection Obj of objects and Arr of arrows. Arrows have a *domain* and *codemain* (or *range*), both in Obj, so a typical arrow is written $f : c \longrightarrow c'$, where c is the domain of f and c' is the codomain of f, and $c, c' \in \mathrm{Obj}(C)$. Arrows can be composed. If $f : c \longrightarrow c'$ and $g : c' \longrightarrow c''$, then there is an arrow $g \circ f : c \longrightarrow c''$. A category satisfies two additional axioms:

1. $(f \circ g) \circ h = f \circ (g \circ h)$
2. $\forall c \in Obj(C) : \exists! \, \mathrm{id}_c : c \longrightarrow c \in \mathrm{Arr}(c) : \quad \forall f : c \longrightarrow c' :$ $\mathrm{id}_{c'} \circ f = f = f \circ \mathrm{id}_c$

Simple example

Another simple example

Let C be a category. Then define the dual category C^* as follows:

- Objects of C^*: same as C
- Arrows of C^*: same as C but with reversed direction

Size matters

Definition

We will denote the class of arrows with domain c and codomain c' by $C(c, c')$, we call these *hom-sets*.

Definition

A category C is called *small* if both $\mathrm{Obj}(C)$ and $\mathrm{Arr}(C)$ are actually sets and not proper classes, and large otherwise. A *locally small category* C is a category such that for all objects c and c' of C, $C(c, c')$ is a set. If $C(c, c')$ is a set, we say it is a *small hom-set* and if C is locally small, we say that it has *small hom-sets*.

Examples

- **Set** Objects: Sets, Arrows: Functions
- **Top** Objects: Topological spaces, Arrows: Continuous functions
- **Met** Objects: Metric spaces, Arrows: Continuous functions
- **Grp** Objects: Groups, Arrows: Group homomorphisms
- **Rng** Objects: Rings, Arrows: Ring homomorphisms
- **Pos** Objects: Posets (partially ordered sets), Arrows: Order preserving functions

Examples

- Can also consider category of abelian groups, compact metric / topological spaces
- More complicated categories like chain complexes, cochain complexes with arrows chain maps

Presheaf (no category theory)

Let X be a topological space, Σ be a collection of sets (not necessarily subsets of X) and denote the collection of open sets of X by $\mathcal{O}(X)$. A *presheaf* F on X is a collection of maps $F : \mathcal{O}(X) \to \Sigma$ which assigns for each open set U of our topological space a set $F(U)$ in Σ. In addition, every time $V \subseteq U$ for open sets in X, we have a map $a(U, V) : F(U) \to F(V)$. We require that these maps satisfy:

1. $a(U, U)$ is the identity map for the set $F(U)$
2. For $W \subseteq V \subseteq U$, then $a(W, V) \circ a(V, U) = a(W, U)$

Presheaf (some category theory)

Let X be a topological space, C be a category and denote the collection of open sets of X by $\mathcal{O}(X)$. A *presheaf F* on X is a collection of maps $F : \mathcal{O}(X) \to \mathrm{Obj}(C)$ which assigns for each open set U of our topological space an object $F(U)$ in our category. In addition, every time $V \subseteq U$ for open sets in X, we have a morphism $a(U, V) : F(U) \to F(V)$ in C. We require that these morphisms satisfy:

1. $a(U, U)$ is the identity morphism for $F(U)$
2. For $W \subseteq V \subseteq U$, then $a(W, V) \circ a(V, U) = a(W, U)$

Monics

Definition

Let C be a category. An arrow $m : c \to c'$ is called *monic* if for each pair of arrows $f : d \to c$, $g : d \to c$, we have $m \circ f = m \circ g$ implies $f = g$.

Example

Consider the category **Grp** of groups and homomorphisms. Then a monomorphism $m : G \to H$ is monic.

Epics

Definition

Let C be a category. An arrow $e : c \longrightarrow c'$ is called *epic* if for each pair of arrows $f : c' \longrightarrow d$, $g : c' \longrightarrow d$, we have $f \circ e = g \circ e$ implies $f = g$.

Example

Consider the category **Grp** of groups and homomorphisms. Then an epimorphism $e : G \longrightarrow H$ is epic.

Isomorphisms

Definition

Let C be a category. A pair of arrows $f : c \to d$ and $g : d \to c$ such that $g \circ f = \mathrm{id}_c$ and $f \circ g = \mathrm{id}_d$ is called an *inverse pair of isomorphisms* and each component is called an *isomorphism*.

Examples

- In **Set**, a bijective function is an isomorphism
- In **Grp**, a group isomorphism is an isomorphism

Definition

Definition

A (covariant) *functor* is a morphism $T : C \to D$ of categories which has an *object function* which assigns to each object c of C an object Tc of B and an *arrow function* which assigns to each arrow $f : c \to c'$ of C an arrow $Tf : Tc \to Tc'$ of D such that $T(1_c) = 1_{Tc}$ and $T(g \circ f) = Tg \circ Tf$ ($\sharp$).

A *contravariant functor* is a functor in which the final condition ($\sharp$) becomes $T(g \circ f) = Tf \circ Tg$.

Simple examples of a functor

- Let C be a category. Define a functor $T : C \longrightarrow C$ by $T(c) = c$ for every $c \in \text{Obj}(C)$ and $T(f) = f$ for every $f \in \text{Arr}(C)$. This is a covariant functor
- Let C be a category and C^* the dual category. Define a functor $T : C \longrightarrow C^*$ by $T(c) = c$ for every $c \in \text{Obj}(C)$ and $T(f) = f^{-1}$ for every $f \in \text{Arr}(C)$. This is a contravariant functor.

Example of a functor

Let D be a category and let $r \in \text{Obj}(D)$ be fixed. We will now construct a slightly more complicated example of a functor $D(r, -) : D \to \textbf{Set}$, which we will call the *hom-functor of D*.

Given an object d from D, $D(r, -)(d)$ returns $D(r, d)$, defined above, and this is an element of **Set**. Given an arrow $f : d \to d'$, $D(r, -)(f)$ returns a function $\bar{f} : D(r, d) \to D(r, d')$, denoted $D(r, f)$, with the property that $D(r, f)(\text{id}_r) = f$, for any arrow $f : r \to r'$.

Example of a functor

If $f : r \to r$ is the identity arrow of r, $D(r, f)$ will be the function $f : D(r, r) \to D(r, r)$ sending each element to itself and for $f, g \in \text{Obj}(D)$, we have $D(r, g \circ f) = D(r, g) \circ D(r, f)$, where the first $\circ$ denotes composition of arrows in the category D and the second denotes composition of functions (which of course are the arrows of **Set**), so that this is a covariant functor.

Definition

Definition

Let C, D be categories. Given functors $S, T : C \to D$, a *natural transformation* $\tau : S \to T$ is a function which assigns to each object c of C an arrow $\tau_c = \tau c : Sc \to Tc$ of D such that for every arrow $f : c \to c'$ of C,

$$Tf \circ \tau_c = \tau_{c'} \circ Sf. \quad \diamond$$

We then say τ_c is *natural in c*. A natural transformation for which every arrow τ_c is invertible ($f : c \to d \Rightarrow \exists g : d \to c$) is called a *natural isomorphism*. For two contravariant functors, $\diamond$ becomes:

$$Sf \circ \tau_{c'} = \tau_c \circ Tf.$$

Advantages

- If we prove a theorem in category theory, it can apply to things seemingly unrelated
- Very general - as mathematicians we like generalising things
- Unifying language to describe many different but related things

Disadvantages

- So far it has mainly been used as a language, not for proving things
- Question: intuitively, what have I been talking about? What is a functor, a monic etc. in pictures / everyday concepts?

Motivation

Group actions and by extension monoid actions have a more intuitive feel to them. This means the above ideas might seem more intuitive if done in different language.

Definition

Definition

Let C be a set equipped with a partial binary operation which we shall denote by $\cdot$ or by concatenation. If $x, y \in C$ and the product $x \cdot y$ is defined we write $\exists x \cdot y$. An element $e \in C$ is called an *identity* if $\exists e \cdot x \Rightarrow e \cdot x = x$ and $\exists x \cdot e \Rightarrow x \cdot e = x$. The set of identities of C is denoted C_0. The pair $(C, \cdot)$ is said to be a *category* if the following axioms hold:

1. $x \cdot (y \cdot z)$ exists if and only if $(x \cdot y) \cdot z$ exists, in which case they are equal
2. $x \cdot (y \cdot z)$ exists if and only if $x \cdot y$ and $y \cdot z$ exist
3. For each $x \in C$ there exist identities e and f such that $\exists x \cdot e$ and $\exists f \cdot x$.

Remarks

Remark

It can be deduced that the identies in (3) are uniquely determined by x (Suppose e and f are both right identities of x. Then $\exists(x \cdot e) \cdot f$. So $\exists e \cdot f$. So $e = e \cdot f = f$). Therefore, we will write $e = \mathbf{d}(x)$ and $f = \mathbf{r}(x)$.

Lemma

$\exists x \cdot y \Leftrightarrow \mathbf{d}(x) = \mathbf{r}(y)$

Proof

Proof.

($\Rightarrow$) Suppose $\exists x \cdot y$. Denote $e := \mathbf{d}(x)$. Then $\exists (x \cdot e) \cdot y$. So $\exists x \cdot (e \cdot y)$. So $\exists e \cdot y$. Since e is an identity, $e \cdot y = y$. Therefore, $e = \mathbf{r}(y)$.

($\Leftarrow$) Suppose $z = \mathbf{d}(x) = \mathbf{r}(y)$. Then $\exists x \cdot z$ and $\exists z \cdot y$. So, $\exists (x \cdot z) \cdot y = x \cdot y$. Therefore, $\exists x \cdot y$. □

Functors

Definition

Let C, D be categories. A *covariant functor* T is a morphism $T : C \to D$ such that:

1. If $e \in C_0$, then $T(e) \in D_0$.
2. If $x \in C$ with $\mathbf{d}(x) = e$ and $\mathbf{r}(x) = f$, then $\mathbf{d}(T(x)) = T(e)$ and $\mathbf{r}(T(x)) = T(f)$.
3. If $\exists x \cdot y$, then $\exists T(x) \cdot T(y)$ and $T(x \cdot y) = T(x) \cdot T(y)$.

A morphism T satisfying (1) & (2), in addition to:

- If $\exists x \cdot y$, then $\exists T(y) \cdot T(x)$ and $T(x \cdot y) = T(y) \cdot T(x)$.

is called a *contravariant functor*.

Natural Transformations

Definition

Let C, D be categories and $S, T : C \to D$ be covariant functors. A *natural transformation* $\tau : S \to T$ is a a function which assigns to each identity $e \in C_0$ an element τ_e of D with $\mathbf{d}(\tau_e) = S(e)$ and $\mathbf{r}(\tau_e) = T(e)$ and for every $y \in C$ with $\mathbf{d}(y) = e$ and $\mathbf{r}(y) = f$ we have $\exists T(y) \cdot \tau_e$, $\exists \tau_f \cdot S(y)$ and

$$T(y) \cdot \tau_e = \tau_f \cdot S(y). \quad \diamond$$

We then say τ_e is *natural* in e.

Natural Transformations

Definition

A natural transformation such that for all such τ_e there is a τ_f with $\mathbf{d}(\tau_f) = T(e)$ and $\mathbf{r}(\tau_f) = S(e)$ is called a *natural isomorphism*. For two contravariant functors, we have the same except $\diamond$ becomes:

$$S(y) \cdot \tau_f = \tau_e \cdot T(y).$$

Actions

Definition

Let C be a category, X a set, and $\mathbf{p} : X \to C_0$ be a function. Let $C * X$ be the set

$$C * X := \{(c, x) \in C \times X : \mathbf{d}(c) = \mathbf{p}(x)\}.$$

We suppose in addition there is a function $C * X \to X$, denoted by $(c, x) \mapsto c \cdot x$. We shall write $\exists c \cdot x$ if $(c, x) \in C * X$. We say that C *acts on* X (on the left), and that X is a *left* C*-system* if the following axioms hold:

1. $\exists\, \mathbf{p}(x) \cdot x$ and $\mathbf{p}(x) \cdot x = x$ for all $x \in X$.
2. If $\exists c \cdot x$, then $\mathbf{p}(c \cdot x) = \mathbf{r}(c)$.
3. If $\exists cd$ in C and $\exists (cd) \cdot x$, then $\exists d \cdot x$ and $\exists c \cdot (d \cdot x)$ and $(cd) \cdot x = c \cdot (d \cdot x)$.