Stiefel-Whitney classes

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1. Singular Cohomology
   - Definition and examples
   - Ring structure

Singular Cohomology

Definition

Given a topological (possibly smooth) manifold M of dimension n, a group G and a chain of modules:

$$\dots \xrightarrow{d} C^{i-1}(M;G) \xrightarrow{d} C^{i}(M;G) \xrightarrow{d} C^{i+1}(M;G) \xrightarrow{d} \dots$$

where $C^i(M;G) = \mathrm{Hom}_G(C_i(M;G), G)$ *for* $C_i(M;G) = \{\sum \sigma_j : \sigma_j : \triangle^i \longrightarrow M \text{ continuous}\}$ *and* $d \circ d = 0$*, we define the* *i-th cohomology group* *of M with coefficients in G:*

$$H^i(M;G) = \frac{\mathrm{Ker}(d : C^i(M;G) \longrightarrow C^{i+1}(M;G))}{\mathrm{Im}(d : C^{i-1}(M;G) \longrightarrow C^i(M;G))}$$

Assume $G = \mathbb{Z}$.

Example

$$H^i(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \textit{if } i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

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We will focus in $G = \mathbb{Z}_2$. This guarantees orientability.

Definition

Given cochains $\psi \in C^m(M)$, $\mu \in C^k(M)$ *their cup product* $\psi \smile \mu \in C^{m+k}(M)$ *is defined by its action on* $\sigma \in C_{m+k}(M)$*:*

$$(\psi \smile \mu)(\sigma) = (-1)^{m+k}\psi(\sigma|_{(t_0,\dots,t_m)}) \cdot \mu(\sigma|_{(t_m,\dots,t_{m+k})})$$

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The cup product turns

$$H(M;G) = \bigoplus_i H^i(M;G)$$

into a graded ring.

Theorem (de Rham)

If M is smooth, there is a ring isomorphism

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Example

The cohomology of the Sphere has ring strucutre

$$H^*(S^n;\mathbb{Z}) = \frac{\mathbb{Z}[a]}{a^2 = 0}, \qquad |a| = 1$$

Example

The Real Projective Space $\mathbb{RP}^n$ has cohomology

$$H^i(\mathbb{RP}^n;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 0,\dots,n \\ 0 & \text{otherwise} \end{cases}$$

and its ring structure is

$$H^*(\mathbb{RP}^n) = \frac{\mathbb{Z}_2[a]}{a^{n+1} = 0}, \qquad |a| = 1$$

Stiefel-Whitney classes and properties

Let $E \longrightarrow M$, $rk(E) = n$, $dim(M) = m$. We will give an axiomatic definition with 4 axioms and compute easy cases. Existance can be proved in general (see Milnor-Stasheff).

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Definition (Stiefel-Whitney classes (I & II))

The Stiefel-Whitney classes of the vector bundle E are

- $$w_i(E) \in H^i(B; \mathbb{Z}_2)$$

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  with $w_i(E) = 0$ for $i > n$ and $w_0(E) = 1$.
- *(Naturality) Given $E \longrightarrow M$, $F \longrightarrow N$ with $f : E \longrightarrow F$ descending to a map on the base spaces:*
  $$w_i(E) = f^* w_i(F)$$

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If E *is trivial then* $w_i(E) = 0$ *for* $i > 0$.

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- *(Whitney Product Theorem) If E and F are vectors over M*

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- *The tautological line bundle over* $\mathbb{RP}^1$*, namely*

$$\gamma_1^1 = \{(x_1 : x_2, v) \in \mathbb{RP}^1 \times \mathbb{R}^2 | v \in (x_1, x_{,2}\}$$

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Lemma

If F is trivial, $w_i(E \bigoplus F) = w_i(E)$.

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Remark

The Whitney Product Theorem can be expressed

$$w(E + F) = w(E)w(F)$$

where the product between elements is the cup product

Lemma

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Proof.

If $w = w_1 + \ldots + w_n$, define its inverse:

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where:

$$\overline{w}_1 = w_1, \qquad \overline{w}_1 = w_1^2 + w_2, \qquad \overline{w}_1 = w_1^3 + w_3$$
$$\overline{w}_n = w_1\overline{w}_{n-1} + w_2 + \overline{w}_{n-2} + \ldots + w_{n-1}\overline{w}_1 + w_n$$

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$E^{\perp}$ is the orthonormal bundle of E.

Computations in $\mathbb{RP}^n$

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Proposition

$$w(\gamma_n^1) = 1 + a$$

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$\tau \oplus \xi^1 \cong \gamma_n^1 \oplus \ldots \oplus \gamma_n^1$ *$(n+1)$ times and therefore:*

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$$w(\mathbb{RP}^n) = w(\tau \oplus \xi^1) = w(\gamma^1_n \oplus \ldots \oplus \gamma^1_n) = (1+a)^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} a^i$$

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As a remark, since all the coefficients of $w(\mathbb{RP}^n) = \sum_{i=0}^n \binom{n+1}{i} a^i$ must be in $\mathbb{Z}_2$, then $w(\mathbb{RP}^7) = w(\mathbb{RP}^3) = 1$.

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Definition

A smooth compact $n-$dimensional manifold M without boundary is nullbordant if it is diffeomorphic to the boundary of some compact smooth $(n+1)-$dimensional manifold W with boundary.

Theorem (Pontrjagin-Thom)

All the Stiefel-Whitney numbers of M are zero if and only if M is nullbordant.

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Challenge! Prove without using this theorem that $\mathbb{RP}^3$ and $\mathbb{RP}^7$ are the boundary of some manifold.

Thanks for coming!